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Boundary Effects in Integrable Field Theory on a Half Line

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Abstract

Boundary effects caused by the boundary interactions in various integrable field theories on a half line are discussed at the classical as well as the quantum level. Only the so-called "integrable" boundary interactions are discussed. They are obtained by the requirement that certain combinations of the lower members of the infinite set of conserved quantities should be preserved. Contrary to the naive expectations, some "integrable" boundary interactions can drastically change the character of the theory. In some cases, for example, the sinh-Gordon theory, the theory becomes ill-defined because of the instability introduced by "integrable" boundary interactions for a certain range of the parameter.

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1 Introduction

Integrable field theory in 1 dimension or on a whole line, both at the classical and quantum levels, have been investigated quite intensively in recent years and many interesting results have been uncovered. The motivation is mainly two-fold: firstly, the integrable theory in its own right as a theoretical laboratory to study the structure of field theory beyond perturbation and secondly its connection with conformal field theory (deformed CFT) and string theory.

In contrast, integrable field theory on a half line, say $x \leq 0$, has a shorter history. Here, the effects of the boundary or the boundary conditions which replace the "asymptotic conditions" in field theory on a whole line are the main objects of research. This problem is also related with integrable statistical lattice models with non-trivial boundary conditions, scattering of electrons by an impurity in solids (the Kondo problem) monopole catalysed baryon decays and deformations of conformal field theory with boundaries. One of the purposes of this paper is to discuss the effects caused by the boundary on the integrable systems. Contrary to the naive expectations, some "integrable" boundary interactions can drastically change the character of the theory. In some cases, for example, the sinh-Gordon theory, the theory becomes ill-defined because of the instability introduced by a specific choice of "integrable" boundary interactions. This will be discussed in some detail in later sections.

There are two types of approaches for the integrable field theory on a half line, algebraic and field theoretical. The algebraic approach was initiated by Cherednik [1] more than ten years ago. Firstly, the dynamical system under consideration is integrable on the whole line having a factorisable S-matrix. Secondly, a natural assumption is that when restricted to the half line, *the particle content (mass spectrum), and the S-matrices describing their mutual interactions, are exactly the same as those on the whole line.* Thirdly, when a particle hits the boundary it is assumed to be reflected elastically (up to rearrangements among mass degenerate particles). The compatibility of the reflections and the scatterings constitutes the main algebraic condition, called Reflection equation [1, 2, 3, 5] which generalises the Yang-Baxter equation. In other words, the effect of the boundary is local and coded into a set of reflection factors $K_{ab}(\theta)$, where a labels the incoming particle, b labels the reflected particle and θ is the rapidity.

In the case of integrable field theory with *diagonal* S-matrices, for example, affine Toda field theory to be discussed in detail in this paper, the Yang-Baxter equation is trivially satisfied. The reflection at the boundary does not change the particle species and the Reflection equation is again trivially satisfied. In this case a new algebraic equation called the Bootstrap equation governs the exact S-matrices and the Reflection equation is replaced by Reflection Bootstrap equation [6, 7], which constrains elastic reflection factors $K_a(\theta)$.

While the algebraic approach is applicable to any theories with exact S-matrices, the field theoretical approach is useful for theories having Lagrangians and classical equations of motion. Our main concern in this paper is to discuss the boundary effects in the field theoretic approach. Here the guiding principle is the infinite set of conserved quantities which guarantees the integrability on a whole line. The boundary potential or the boundary interaction is so chosen as to preserve the set of conserved quantities or its suitable subset. Then the boundary effects can be deduced from the explicit forms of the boundary interactions and field theoretical methods at the classical and/or the quantum levels. For well known integrable field theories on a whole line, sine-Gordon, non-linear Schrödinger and affine Toda field theories, the "integrable" boundary interactions are deduced and/or conjectured [3, 4, 8, 9]. It should be emphasised, however, that not all of these "integrable" boundary interactions give an integrable field theory on a half line. Namely, preserving some infinite subset of conserved quantities is a necessary but not

sufficient condition. Compatibility with the other principles of field theory, in particular, at the quantum level, must be checked carefully.

We will discuss the boundary effects in integrable field theory on a half line mostly taking explicit examples from affine Toda field theories. This is because affine Toda field theory is one of the best understood integrable field theory both at the classical and quantum levels. The algebraic as well as the field theoretical approaches to affine Toda field theory on a whole line have been very successful. This paper is organised as follows: in section 2 affine Toda field theory is briefly reviewed in order to set the stage and to introduce notation. In section 3 very simple examples of a harmonic oscillator with a negative spring constant and a 1 dimensional wave (string) with "integrable" boundary interactions are discussed. It is shown that the instability caused by a special "integrable" boundary interaction makes the theory ill-defined. In section 4 the boundary effects in the simplest and best known affine Toda field theories, the sinh-Gordon and sine-Gordon theories are discussed. Here it is shown that for a certain range of the parameter in the "integrable" boundary interaction the theory possesses instability. In section 5 the boundary effects in affine Toda field theory, mainly the a_n series are discussed. Section 6 is for summary and discussion.

2 Affine Toda field theory on a half line

Affine Toda field theory [10] is a massive scalar field theory with exponential interactions in $1+1$ dimensions described by the action

$$S=\int dt\int_{-\infty}^{\infty}dx\mathcal{L}. \tag{2.1}$$

Here the Lagrangian density is given by

$$\mathcal{L}=\frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a-V(\phi) \tag{2.2}$$

in which

$$V(\phi)=\frac{m^2}{\beta^2}\sum_0^r n_i e^{\beta\alpha_i\cdot\phi}. \tag{2.3}$$

Here ϕ is an r-component scalar field, r is the rank of a compact semi-simple Lie algebra g with α_i; $i=1,\ldots,r$ being its simple roots. The roots are normalised so that long roots have length 2, $\alpha_L^2=2$. An additional root, $\alpha_0=-\sum_1^r n_i\alpha_i$ is an integer linear combination of the simple roots, is called the affine root; it corresponds to the extra spot on an extended Dynkin-Kac diagram for $\hat{g}$ and $n_0=1$. When the term containing the extra root is removed, the theory becomes conformally invariant (conformal Toda field theory). The simplest affine Toda field theory, based on the simplest Lie algebra a_1, the algebra of $su(2)$, is called sinh-Gordon theory, a cousin of the well known sine-Gordon theory. m is a real parameter setting the mass scale of the theory and β is a real coupling constant, which is relevant only in quantum theory.

Toda field theory is integrable at the classical level due to the presence of an infinite number of conserved quantities. Many beautiful properties of Toda field theory, both at the classical and quantum levels, have been uncovered in recent years. In particular, it is firmly believed that the integrability survives quantisation. The exact quantum S-matrices are known [11, 12, 13, 14, 15][16, 17] for all the Toda field theories based on non-simply laced algebras as well as those based on simply laced algebras. The singularity

structure of the latter S-matrices, which in some cases contain poles up to 12-th order [13], is beautifully explained in terms of the singularities of the corresponding Feynman diagrams [18], so called Landau singularities.

For the theory on a half line, (2.1) will be replaced by

$$S = \int dt \left[\int_{-\infty}^{0} \mathcal{L} dx - \mathcal{B} \right], \tag{2.4}$$

where $\mathcal{B}$, a function of the fields but not their derivatives, represents the boundary interaction. The stationarity condition of the action implies the equation of motion which is the same as on the whole line and the boundary condition at $x = 0$

$$\frac{\partial \phi}{\partial x} = -\frac{\partial \mathcal{B}}{\partial \phi}. \tag{2.5}$$

For any choice of $\mathcal{B}$, the energy

$$\mathcal{E} = \int_{-\infty}^{0} \left[\frac{1}{2}(\partial_t \phi^a)^2 + \frac{1}{2}(\partial_x \phi^a)^2 + V(\phi) \right] dx + \mathcal{B} \tag{2.6}$$

is always conserved. But it is *no longer positive definite* for negative boundary interaction ($\mathcal{B} < 0$).

In [8, 9] it was conjectured based on the analysis of low spin conserved quantities that the generic form of the "integrable" boundary interaction is given by

$$\mathcal{B} = \frac{m}{\beta^2} \sum_{0}^{r} A_i e^{\frac{\beta}{2} \alpha_i \cdot \phi}, \tag{2.7}$$

where the coefficients A_i, $i = 0, \ldots, r$ are a set of real numbers. The condition (2.7) is a generalisation of the well known results for the sine-Gordon theory ($r = 1$), in which case the coefficients A_1 and A_0 are completely arbitrary. However, for the affine Toda field theory for higher rank algebras ($r \geq 2$), the coefficients are severely constrained due to the presence of higher spin conserved quantities. For example, for the affine Toda field theories based upon the $a_n^{(1)}$ series of Lie algebras the sequence of conserved charges includes all spins (except zero) modulo $n + 1$. Except for $a_1^{(1)}$, which corresponds to the sinh-Gordon theory, each of these theories has conserved charges of spin ± 2. It was shown in [8] that a combination of spin ± 2 conserved charges as well as spin ± 3 conserved charges are preserved in the presence of the boundary interaction if the boundary interaction term has the form (2.7) with the further constraint:

$$\textbf{either } A_i = \pm 2, \text{ for } i = 0, \ldots, n \textbf{ or } A_i = 0 \text{ for } i = 0, \ldots, n\ . \tag{2.8}$$

By similar analysis, a more general conjecture is obtained which applies to all simply-laced affine Toda field theories [9, 19]

$$\textbf{either } A_i = \pm 2\sqrt{n_i}, \text{ for } i = 0, \ldots, r \textbf{ or } A_i = 0 \text{ for } i = 0, \ldots, r\ . \tag{2.9}$$

The sine- and sinh-Gordon theories and possibly the Bullough-Dodd theory (based on the $a_2^{(2)}$ algebra) and other non-simply laced theories [19, 20] seem to be the only ones for which there is a continuum of possible "integrable" boundary interactions. We will show in section 4 that some part of the continuum might not be realised in field theory. For the others, the possible "integrable" boundary interactions consist of a choice of signs.

3 Simple examples of instability caused by boundary

Let us start with a very simple example of a dynamical system with one degree of freedom, namely a harmonic oscillator with an arbitrary spring constant k, which is either *positive* or *negative*

$$L = \frac{1}{2}(\frac{dy(t)}{dt})^2 - \frac{1}{2}k(y(t))^2. \tag{3.1}$$

In either case, the system has one conserved quantity, the energy

$$\mathcal{E} = \frac{1}{2}(\frac{dy(t)}{dt})^2 + \frac{1}{2}k(y(t))^2, \tag{3.2}$$

and satisfies the formal criterion of "integrability". The solutions of the equation of motion

$$\frac{d^2y}{dt^2} = -ky \tag{3.3}$$

are oscillatory for *positive* spring constant but they grow or decrease exponentially for *negative* spring constant k

$$y(t) = \pm e^{\pm\omega(t-t_0)}, \quad \omega^2 = -k, \quad \text{for} \;\; k < 0; \quad t_0 \;\; \text{arbitrary}. \tag{3.4}$$

Namely the system is unstable for negative k and hardly qualifies to be called integrable in spite of the existence of the conserved energy. In this case the energy is *no longer positive definite* and fails to constrain the system. In fact it is easy to see that these unstable exponential solutions have *zero energy.* The existence of non-trivial zero energy solutions (vacuum solutions) would imply that the system can undergo certain changes without costing energy, almost synonymous to instability. A *negative energy solution* is also easily obtained

$$y(t) = e^{x_0}\cosh\omega t, \quad x_0 \;\; \text{arbitrary}. \tag{3.5}$$

The energy can be made as large and negative as desired by choosing large x_0. A quantum version of such systems, if any, would have serious difficulties.

Next let us discuss the simplest integrable field theory on a half line, namely a massless field (string) with a quadratic boundary potential,

$$\mathcal{L} = \frac{1}{2}\left[(\frac{\partial\phi(x,t)}{\partial t})^2 - (\frac{\partial\phi(x,t)}{\partial x})^2\right], \quad \mathcal{B} = \frac{1}{2}A\phi^2(0,t). \tag{3.6}$$

The equation of motion is the ordinary wave equation

$$\partial_t^2\phi - \partial_x^2\phi = 0 \tag{3.7}$$

and the boundary condition is

$$\frac{\partial\phi(x,t)}{\partial x}\Big|_{x=0} = -A\phi(0,t). \tag{3.8}$$

This is a linear system, with a conserved energy

$$\mathcal{E} = \int_{-\infty}^{0}\left[\frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\partial_x\phi)^2\right]dx + \frac{1}{2}A\phi^2(0,t). \tag{3.9}$$

Therefore it is "integrable" in the same sense as the harmonic oscillator.

The meaning of the boundary interaction is now clear. It attaches a spring with spring constant A, which is positive or negative, to the endpoint of the string. If $A<0$ the energy is *no longer positive definite* and unstable solutions exist:

$$\phi(x,t)=\pm e^{-A(x\pm t-x_0)}, \quad \text{for} \quad A<0. \tag{3.10}$$

They are finite everywhere on the half line $x\leq 0$ for finite time t and localised near the boundary. It is again very easy to see that they have *zero energy*. A *negative energy solution* is easily obtained

$$\phi(x,t)=e^{-A(x-x_0)}\cosh At, \quad x_0 \text{ arbitrary}. \tag{3.11}$$

It is easy to find a 'plane wave' basis satisfying the boundary condition

$$u_p(x)\propto (ip-A)e^{ipx}+(ip+A)e^{-ipx}. \tag{3.12}$$

It is also elementary to show that the above 'plane wave' basis is orthogonal to the localised solution e^{-Ax} for $A<0$. This means that the 'plane wave' basis is not complete for $A<0$. Therefore the initial value problem

$$\phi(x,t=0)=F(x), \quad \partial_t\phi(x,t=0)=G(x), \quad x\leq 0 \tag{3.13}$$

for the string on the half line with the boundary is *unstable* for $A<0$ unless $F(x)$ and $G(x)$ are exactly orthogonal to the localised solution e^{-Ax}. A quantum version of such a theory, if any, would meet serious difficulties. This simple example shows clearly that some boundary effects, even if they are "integrable", are not "local" and can make the theory ill-defined by the instability.

One can easily get the reflection factor [8, 5] of the boundary from the 'plane wave' basis

$$K(p)=\frac{ip+A}{ip-A} \tag{3.14}$$

for either sign of A. But for $A<0$ such a result seems superficial because of the instability of the theory.

It is easy to note that adding a mass to the field tends to stabilise the theory, as the mass term simply gives an attractive harmonic potential with a spring constant m^2 at each point:

$$\mathcal{L}=\frac{1}{2}\left[(\partial_t\phi(x,t))^2-(\partial_x\phi(x,t))^2-m^2\phi(x,t)^2\right], \quad \mathcal{B}=\frac{1}{2}A\phi^2(0,t). \tag{3.15}$$

Therefore the *zero energy* unstable mode exists only for $A\leq -m$

$$\phi(x,t)=e^{\pm\omega t-A(x-x_0)}, \quad \omega^2=A^2-m^2. \tag{3.16}$$

The latter condition is a disguise of the mass shell condition. A *negative energy solution* is given by

$$\phi(x,t)=\cosh\omega t\, e^{-A(x-x_0)}, \tag{3.17}$$

and the energy is a function of x_0, $\mathcal{E}=\frac{\omega^2}{4A}e^{2Ax_0}$, which can be as large and negative as $x_0\to-\infty$. The 'plane wave' basis and the reflection factor have the same form as the wave case.

4 Sinh- and Sine-Gordon theories

Next let us consider sinh-Gordon theory, the simplest member of the affine Toda field theories, based on the $a_1^{(1)}$ algebra. The Lagrangian density and the boundary interactions in the notation of (2.2) are

$$\mathcal{L} = \frac{1}{2}\left[(\partial_t\phi)^2 - (\partial_x\phi)^2\right] - \frac{m^2}{\beta^2}\left(e^{\sqrt{2}\beta\phi} + e^{-\sqrt{2}\beta\phi}\right),\ \mathcal{B} = \frac{mA}{\beta^2}\left(e^{\beta\phi/\sqrt{2}} + e^{-\beta\phi/\sqrt{2}}\right). \quad (4.1)$$

Here the parameter A is arbitrary. Since the exponential functions are always positive, it is expected that the boundary interaction term would induce strong instability for large and negative A. We show this by constructing explicit classical solutions for the equation of motion

$$\partial_t^2\phi - \partial_x^2\phi = -2\sqrt{2}\frac{m^2}{\beta}\sinh\sqrt{2}\beta\phi, \quad (4.2)$$

with the boundary condition

$$\partial_x\phi\Big|_{x=0} = -\sqrt{2}\frac{mA}{\beta}\sinh\beta\phi/\sqrt{2}\Big|_{x=0}. \quad (4.3)$$

It is elementary to see that for $A < 0$

$$\tanh\frac{\sqrt{2}\beta\phi(x,t)}{4} = e^{\pm\omega t}e^{-mA(x-x_0)}, \quad x_0 > 0 \quad (4.4)$$

are solutions provided

$$\omega^2 = m^2(A^2 - 4) \geq 0. \quad (4.5)$$

For the positive sign, the r.h.s. of (4.4) eventually exceeds 1, which is not allowed for tanh function with real arguments.

Namely, for $A \leq -2$ they are real and unstable solutions with one arbitrary parameter $x_0 > 0$. It is also elementary to check that they are *zero energy solutions*. For this the conserved energy takes the form

$$\begin{aligned}\mathcal{E} \;&=\; \int_{-\infty}^{0}\left[\frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\partial_x\phi)^2 + \frac{m^2}{\beta^2}\left(e^{\sqrt{2}\beta\phi} + e^{-\sqrt{2}\beta\phi} - 2\right)\right]dx \\ &\quad + \frac{mA}{\beta^2}\left(e^{\beta\phi/\sqrt{2}} + e^{-\beta\phi/\sqrt{2}} - 2\right)\Big|_{x=0},\end{aligned} \quad (4.6)$$

in which constants are adjusted so that the trivial "vacuum solution" $\phi = 0$ has zero energy. This analysis also clarifies the dynamical meaning of the $A = -2$ condition (2.8), which is the critical point for the instability. It should be remarked that for $A = -2$ there exists a one-parameter (x_0) family of *time independent zero energy solutions* which could be interpreted as degenerate vacuua. There are also *negative energy solutions* for $A < -2$. They are obtained by solving an initial value problem ($x_0 > 0$)

$$\phi(x, t = 0) = \frac{4}{\sqrt{2}\beta}\text{arctanh}e^{-mA(x-x_0)}, \quad \partial_t\phi(x, t = 0) = 0. \quad (4.7)$$

It is easy to calculate the energy

$$\mathcal{E} = \frac{2m}{\beta^2}(A - \frac{4}{A})\frac{a}{1-a}, \quad a = e^{2mAx_0}, \tag{4.8}$$

which goes to $-\infty$ as $x_0 \to 0$. In contrast, energy is bounded for $A \geq -2$ [9].

Since sinh- and sine-Gordon theories are closely related, it is expected that sine-Gordon theory with "integrable" boundary interaction has instability for certain range of the parameter A. The Lagrangian density and the boundary interactions for sine-Gordon theory are

$$\mathcal{L} = \frac{1}{2}\left[(\partial_t\phi)^2 - (\partial_x\phi)^2\right] - \frac{2m^2}{\beta^2}\left(1 - \cos\sqrt{2}\beta\phi\right), \quad \mathcal{B} = \frac{2mA}{\beta^2}\left(1 - \cos\beta\phi/\sqrt{2}\right). \tag{4.9}$$

Here the parameter A is arbitrary. We have added constant terms to the potential and boundary interaction term in such a way that they vanish for the trivial "vacuum solution" $\phi = 0$. This also makes them positive definite (up to the sign of A). Since the cosine is a bounded function, it is expected that the instability caused by the boundary interaction term, if any, would be milder than the sinh-Gordon case. We show this by constructing explicit classical solutions for the equation of motion

$$\partial_t^2\phi - \partial_x^2\phi = -2\sqrt{2}\frac{m^2}{\beta}\sin\sqrt{2}\beta\phi, \tag{4.10}$$

with the boundary condition

$$\partial_x\phi\Big|_{x=0} = -\sqrt{2}\frac{mA}{\beta}\sin\beta\phi/\sqrt{2}\Big|_{x=0}, \tag{4.11}$$

in the same way as in the sinh-Gordon theory. It is elementary to see that

$$\tan\frac{\sqrt{2}\beta\phi(x,t)}{4} = e^{\pm\omega t}e^{-mA(x-x_0)}, \quad x_0 \ \text{ arbitrary} \tag{4.12}$$

are solutions provided

$$\omega^2 = m^2(A^2 - 4) \geq 0. \tag{4.13}$$

Namely, for $A^2 \geq 4$ they are ordinary kink solutions with a specific x dependence. The one arbitrary parameter x_0 is interpreted as the kink position. In sharp contrast with the sinh-Gordon case, the field ϕ is always finite for finite x and t. However, for $A \leq -2$ (negative boundary term) they are *zero energy solutions*, a sign of instability. For this the conserved energy takes the form

$$\begin{aligned}\mathcal{E} \quad &= \quad \int_{-\infty}^{0}\left[\frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\partial_x\phi)^2 + \frac{2m^2}{\beta^2}\left(1 - \cos\sqrt{2}\beta\phi\right)\right]dx \\ &\quad + \frac{2mA}{\beta^2}\left(1 - \cos\beta\phi/\sqrt{2}\right)\Big|_{x=0} \geq \frac{4mA}{\beta^2},\end{aligned} \tag{4.14}$$

in which constants are adjusted so that the trivial "vacuum solution" $\phi = 0$ has zero energy. As in the case of sinh-Gordon theory, for $A = -2$ there exists a one-parameter (x_0)

family of *time independent zero energy solutions* which could be interpreted as degenerate vacuua. Negative energy solutions can also be obtained by solving an initial value problem similar to (4.7). However, in contrast with the sinh-Gordon theory, the energy is bounded from below.

Let us conclude this section by remarking that field theoretical formulation of sinh- and/or sine-Gordon theories with "integrable" boundary interaction for $A \le -2$ both at the classical and quantum levels seems problematic. It might be possible to interpret the plethora of the solutions of Reflection Bootstrap equation [7] in terms of the instability. Further investigation is definitely wanted.

5 $a_n^{(1)}$ Toda field theory

Among the 2^{n+1} possible choices of the "integrable" boundaries for $a_n^{(1)}$ Toda field theory (2.8), here we discuss only two cases, namely all A_i are the same

$$A_i = A = \pm 2, \quad i = 0, 1, \ldots, n. \tag{5.1}$$

The other cases, with mixed pluses and minuses, break the Z_{n+1} symmetry which is essential for the determination of the bulk S-matrices [11, 13]. Moreover, in these cases, the "vacuum" $\phi = 0$ is no longer the solution of equation of motion and the boundary condition. Therefore, the "new vacuum", if any, of such theories is different from that of the whole line. The particle spectrum and their interactions are also different due to the lack of Z_{n+1} symmetry and the different vacuum.

First we show that $a_{2n+1}^{(1)}$ theory with $A = -2$ "integrable" boundary interaction is *unstable* in spite of the Z_{2n+2} symmetry. We have shown this for $a_1^{(1)}$ theory, the sinh-Gordon case. $a_{2n+1}^{(1)}$ theory with $A = -2$ has also a one parameter (x_0) family of *zero energy solutions* (degenerate vacuua) of the same form as in the sinh-Gordon theory:

$$\begin{aligned} \phi(x,t) &= \mu\varphi(x,t), \quad \mu = \alpha_1 + \alpha_3 + \cdots + \alpha_{2n+1}, \\ \tanh\frac{\beta\varphi(x,t)}{2} &= e^{-A(x-x_0)}. \end{aligned} \tag{5.2}$$

The single component field φ in the special direction μ satisfies all the $2n+1$ equations for ϕ. One only has to note $\alpha_i \cdot \mu = 2$ for i odd and -2 for i even. This phenomenon is called "dimension one reduction" and has been discussed in some detail in [21] for various types of Toda field theory. However, it should be remarked that the reductions applicable to the whole line are not guaranteed to work for the boundary. The existence of the zero (frequency) mode was also noted in [8] within the linear approximation. It should be remarked that the above solution can also be obtained by using Hirota's method.

In contrast, $a_{\text{even}}^{(1)}$ theory with $A = -2$ seems to have no degenerate vacuua. So it deserves further investigation. Let us consider the weak coupling limit ($\beta \ll 1$), or the linear approximation. Then the system is a sum of independent Klein-Gordon fields, with "integrable" quadratic boundary interactions:

$$\begin{aligned} \mathcal{L} &= \sum_{j=1}^{2n} |\partial_\mu\phi_j|^2 - m_j^2|\phi_j|^2, \quad m_j = 2m\sin\frac{j\pi}{2n+1}, \\ \mathcal{B} &= -\frac{1}{2m}\sum_{j=1}^{2n} m_j^2|\phi_j|^2. \end{aligned} \tag{5.3}$$

Due to the attractive boundary effect, each field has a localised solution, $e^{\frac{m_j^2}{2m}x}$, namely a boundary bound state. As before this state is orthogonal to the 'plane wave' basis

$$u_p^{(j)}(x) \propto (ip + \frac{m_j^2}{2m})e^{ipx} + (ip - \frac{m_j^2}{2m})e^{-ipx}.$$

This means that upon quantisation, one needs to introduce the creation and annihilation operators for the 'plane wave' states as well as the boundary bound states:

$$\begin{aligned}\phi_j(x,t) &= \frac{1}{2\pi}\int_0^\infty \frac{dp}{\sqrt{N_p}}\left\{e^{-i\omega_p t}a_j(p) + e^{i\omega_p t}a_{\bar{j}}^\dagger(p)\right\} u_p^{(j)}(x) \\ &+ \frac{1}{\sqrt{N_j}}\left\{e^{-i\omega_j t}b_j + e^{i\omega_j t}b_{\bar{j}}^\dagger\right\} e^{\frac{m_j^2}{2m}x}. \end{aligned} \tag{5.4}$$

Here N_p and N_j are normalisation constants and $\omega_p^2 = p^2 + m_j^2$, $\omega_j^2 = m_j^2(1 - \frac{m_j^2}{4m^2})$. Otherwise the Heisenberg commutation relations

$$[\phi_j(x,t), \partial_t\phi_k(y,t)] = i\delta_{j\bar{k}}\delta(x-y)$$

cannot be satisfied. Namely the spectrum is changed by the boundary. Therefore a naive correspondence with the algebraic approach à la Cherednik seems to be lost. However, it is tempting to relate the emergence of the boundary bound states with the plethora of the solutions of the Reflection Bootstrap equation. Certain characteristic differences between the solutions of reflection Bootstrap equation for a_{even} and a_{odd} theories are also reported [7].

6 Summary and discussion

The effects of the 'boundary interactions' are analysed for various types of integrable field theories on a half line. Here we discuss only the "integrable" boundary interactions such that they preserve certain subset of the classical infinite set of conserved quantities. It is shown that not all of these 'integrable' boundary interactions give consistent quantum field theories on a half line. Especially when the boundary interaction is negative and strong, some theories become ill-defined due to the instabilities which are related with the non-positive definite energy.

For sinh-Gordon theory with a strong negative boundary interaction, an explicit 1-parameter (x_0) family of unstable solutions (4.4) are constructed, which has the form of 1-soliton solution with x_0 being its position. It is well known that sinh-Gordon theory and other affine Toda field theories on a whole line have no soliton solutions. The negative boundary interaction enables the soliton solution and thereby causes the instability. It is also interesting that the unstable solution can be obtained from the "trivial vacuum" $\phi = 0$ by the Bäcklund Transformation (B-T). The close similarity between the B-T and the "integrable" boundary interactions [4, 22] will be discussed elsewhere [23].

Although we did not discuss the free boundary, $\mathcal{B} = 0$, in this paper, this does not mean the free boundary is uninteresting. The free boundary always preserves the necessary conserved quantities and therefore is "integrable" , not only for affine Toda field theory but also for other types of theories like non-linear sigma models which have non-diagonal

S-matrices. The corresponding Neumann boundary condition, $\partial_x\phi = 0$ at $x = 0$ is always satisfied if ϕ is extended to the whole line as an even function. It is an interesting challenge to derive reflection factors $K_a(\theta)$ corresponding to the free boundary for various models in terms of the field theoretical methods.

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References

[1] I. V. Cherednik, *Theor. Math. Phys.* **61** (1984) 977.

[2] S. Ghoshal and A. B. Zamolodchikov, *Int. J. Mod. Phys.* **A9** (1994) 3841; S. Ghoshal, *Int. J. Mod. Phys.* **A9** (1994) 4801.

[3] E. K. Sklyanin, *Funct. Analy. Appl.* **21** (1987) 164; *J. Phys.* **A21** (1988) 2375.

[4] V. O. Tarasov, *Inverse Problems* **7** (1991) 435.

[5] M. T. Grisaru, L. Mezincescu and R. I. Nepomechie, hep-th/9407089.

[6] A. Fring and R. Köberle, *Nucl. Phys.* **B421** (1994) 159; *Nucl. Phys.* **B419** (1994) 647.

[7] R. Sasaki, In "Interface Between Physics and Mathematics" eds. W. Nahm and J-M. Shen, World Scientific (1994) 201; hep-th/9311027.

[8] E. Corrigan, P. E. Dorey, R. H. Rietdijk and R. Sasaki, *Phys. Lett.* **B333** (1994) 83.

[9] E. Corrigan, P. E. Dorey, R. H. Rietdijk, Supplement of Prog. Theor. Phys. **118** (1995).

[10] A. V. Mikhailov, M. A. Olshanetsky and A. M. Perelomov, *Comm. Math. Phys.* **79** (1981) 473;
D. I. Olive and N. Turok, *Nucl. Phys.* **B265** (1986) 469.

[11] A. E. Arinshtein, V. A. Fateev, A. B. Zamolodchikov, *Phys. Lett.* **B87** (1979) 389.

[12] H. W. Braden, E. Corrigan, P. E. Dorey and R. Sasaki, *Phys. Lett.* **B227** (1989) 411.

[13] H. W. Braden, E. Corrigan, P. E. Dorey and R. Sasaki, *Nucl. Phys.* **B338** (1990) 689; H. W. Braden and R. Sasaki, *Phys. Lett.* **B255** (1991) 343; R. Sasaki and F. P. Zen, *Int. J. Mod. Phys.* **8** (1993) 115.

[14] P. Christe and G. Mussardo, *Nucl. Phys.***B330**(1990) 465; *Int. J. Mod. Phys.* **A5** (1990) 4581.

[15] C. Destri and H. J. de Vega, *Phys. Lett.* **B233** (1989) 336.

[16] G. W. Delius, M. T. Grisaru, D. Zanon, *Nucl. Phys.* **B382** (1992) 365.

[17] E. Corrigan, P. E. Dorey and R. Sasaki, *Nucl. Phys.* **B408** (1993) 579.

[18] H. W. Braden, E. Corrigan, P. E. Dorey and R. Sasaki, *Nucl. Phys.* **B356** (1991) 469.

[19] P. Bowcock, E. Corrigan, P. E. Dorey and R. H. Rietdijk, DTP-94/57, hep-th/9501098; S. Penati and D. Zanon, IFUM-490-FT, hep-th/9501105.

[20] A. Fujii, unpublished.

[21] R. Sasaki, *Nucl. Phys.* **B383** (1992) 291.

[22] H. Saleur, S. Skorik and N. P. Warner, USC-94-013 hep-th/9408004.

[23] A. Fujii and R. Sasaki, in preparation.